Lecture 16

4.6 Phase Portraits

Given Hamiltonian $H(\underline{q},\underline{p})$, we have Hamilton's equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad i \in \{1,\dots,N\}$$

Remark: If $\dot{q}_i = 0$, $\dot{p}_i = 0$, $i = 1,\dots,N$, then the system is at rest

Then we have

$$\frac{\partial H}{\partial q_i} = 0 \text{ and } \frac{\partial H}{\partial p_i} = 0, \quad i = 1,\dots,N$$ ← defines critical points of H

Then investigate their nature

- local minimum - stable fixed point
- local maximum - unstable fixed point
- saddle point - mixture

'Plot' these and investigate dynamics in their neighbourhoods

→ qualitative picture (phase portrait)

Example: (1)

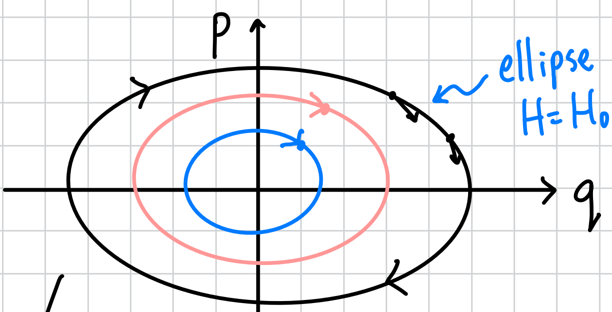
$$H = \frac{p^2}{2m} + \frac{\omega^2 q^2}{2}, \quad \omega, m > 0 \text{ and constant}$$

The critical points are

$$\frac{\partial H}{\partial p} = \frac{p}{m} = 0, \quad \frac{\partial H}{\partial q} = \omega^2 q = 0 \quad \text{at } (q,p) = (0,0)$$

$$\dot{q} = \frac{p}{m} \qquad \dot{p} = -\omega^2 q$$

$$q(0) = q_0 \qquad p(0) = p_0$$



$$H_0 = \frac{p_0^2}{2m} + \frac{\omega^2 q_0^2}{2} = \frac{p^2}{2m} + \frac{\omega^2 q^2}{2}$$ (As H is conserved in each motion)

We get a collection of ellipses labelled by H

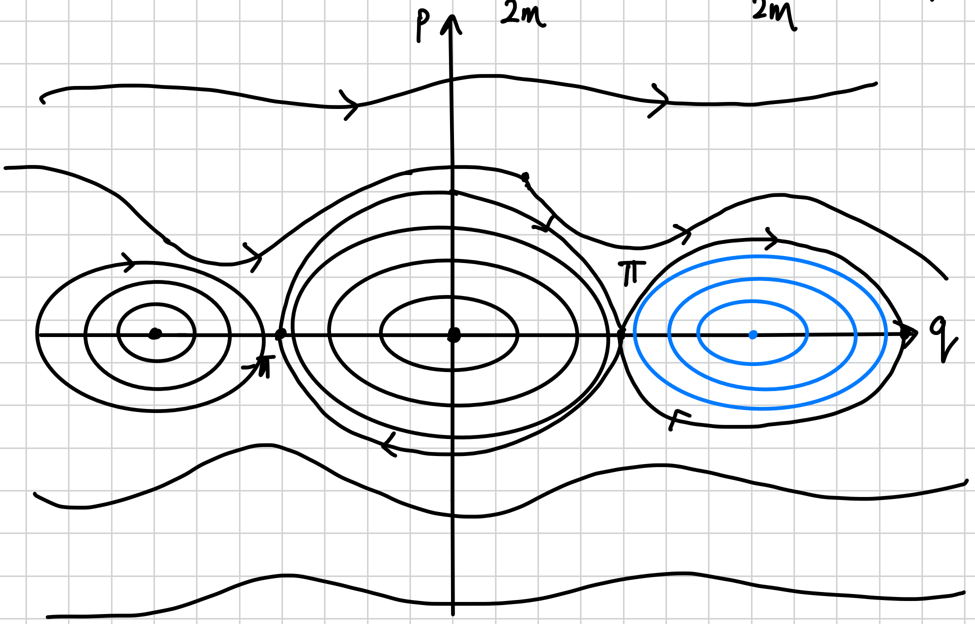
Example (2) Take Hamiltonian

$$H(q,p) = \frac{p^2}{2m} - w^2 \cos q \quad , \quad m, w > 0 \quad \text{constant}$$

critical points $\quad \frac{\partial H}{\partial p} = \frac{p}{m} \; , \; \frac{\partial H}{\partial q} = w^2 \sin q \; , \; \frac{\partial H}{\partial p} = 0, \; \frac{\partial H}{\partial q} = 0$

$$\Rightarrow (q,p) = (0,0), \; (k\pi, 0) \qquad k = 0, \pm 1, \pm 2 \cdots$$

H is still constant, $\quad H_0 = \frac{p_0^2}{2m} - w^2 \cos q_0 = \frac{p^2}{2m} - w^2 \cos q$



?

4.7 Symmetries and Conserved Quantities

Suppose we have a system described by a Lagrangian $\mathcal{L}(q_i, \dot{q}_i)$

Also suppose that $\mathcal{L}$ is invariant under some change of Variables

For a "small" (infinitesmal) transformation,

$$q_i \to q_i + \delta q_i , \quad \dot{q}_i \to \dot{q}_i + \delta \dot{q}_i$$

$$\mathcal{L}(q_i + \delta q_i , \dot{q}_i + \delta \dot{q}_i) = \mathcal{L}(q_i, \dot{q}_i) + \sum_{i=1}^{N} \delta q_i \frac{\partial \mathcal{L}}{\partial q_i} + \sum_{i=1}^{N} \delta \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

Multivariable Taylor

$$\parallel$$

$$\mathcal{L}(q_i, \dot{q}_i)$$

$$\Rightarrow \sum_{i=1}^{N} \delta q_i \frac{\partial \mathcal{L}}{\partial q_i} + \sum_{i=1}^{N} \delta \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0$$

$$\sum_{i=1}^{N} \frac{d}{dt}\left(\delta q_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \sum_{i=1}^{N} \delta q_i \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)$$

Use Euler-Lagrange eqn to get

$$\frac{d}{dt}\sum_{i=1}^{N}\left(\delta_{q_i}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)=0 \Rightarrow \sum_{i=1}^{N}\delta q_i\, p_i \text{ is conserved}$$

Example: Particle of mass m in 3D (represented by $\underline{r}$), subject to a force given by

$$\underline{F}=-\underline{\nabla}V(|\underline{r}|)$$

Then the Lagrangian is

$$\mathcal{L}=\frac{1}{2}m|\dot{\underline{r}}|^2-V(|\underline{r}|)$$

If $\underline{r}=\sum_{i=1}^{n}x_i\underline{e}_i$, $\underline{e}_i$ is fixed, x_i, $i=1,2,3$ are usual co-ordinates

↳ This is invariant under rotation

$\Rightarrow \mathcal{L}$ is invariant under changes of variable

$$x_i \to x_i'=R_{ij}x_j\,,\quad RR^T=\mathbb{1}$$

To see what is conserved, look at an infinitesimal rotation,

$$(\mathbb{1}+\delta R)(\mathbb{1}+\delta R^T)=\mathbb{1} \Rightarrow \delta R+\delta R^T=0 \quad \text{(first order)}$$

$$\Rightarrow \delta R=-\delta R^T \quad \text{(antisymmetric)}$$

and therefore we can write

$$(\delta R)_{ij}=\varepsilon_{ijk}\delta\omega_k$$

Hence

$$\delta x_i=\delta R_{ij}x_j=\varepsilon_{ijk}\omega_k x_j$$

Then the conserved quantity is

$$\varepsilon_{ijk}\delta\omega_k x_j p_i=\varepsilon_{ijk}x_i p_i\delta\omega_k$$
$$=-(\underline{r}\times\underline{p})\cdot\delta\underline{\omega}$$

$\Rightarrow (\underline{r}\times\underline{p})$ is conserved $\Rightarrow$ angular momentum conserved

↑

angular momentum

Remark: The Hamiltonian for the system is

$$H = \frac{|\underline{p}|^2}{2m} + V(\underline{r})$$

Compute $J_a = \varepsilon_{abc} x_b p_c$ and then $\{J_a, H\}$

$$\{J_a, H\} = \sum_d \left(\frac{\partial J_a}{\partial x_d} \frac{\partial H}{\partial p_d} - \frac{\partial J_a}{\partial p_d} \frac{\partial H}{\partial x_d} \right)$$

$$= \varepsilon_{abc} \delta_{bd} \frac{p_d}{m} p_c - \varepsilon_{abc} x_c \delta_{ad} \frac{x_d}{|\underline{r}|} V'(|\underline{r}|)$$

$$= \varepsilon_{adc} \frac{p_d p_c}{m} - \varepsilon_{dbc} x_c \frac{x_d}{|\underline{r}|} V'(|\underline{r}|)$$

$$= 0 \ ?$$

Remark: Compute $\{J_1, J_2\}$

$$J_1 = \varepsilon_{1bc} x_b p_c = x_2 p_3 - x_3 p_2$$

$$J_2 = \varepsilon_{2bc} x_b p_c = x_3 p_1 - x_1 p_3$$

$$\{J_1, J_2\} = \frac{\partial}{\partial x_d}(x_2 p_3 - x_3 p_2) \frac{\partial}{\partial p_d}(x_3 p_1 - x_1 p_3) - \frac{\partial}{\partial p_d}(x_2 p_3 - x_3 p_2) \frac{\partial}{\partial x_d}(x_3 p_1 - x_1 p_3)$$

$$= (-p_3)(-x_1) - (x_2) p_1$$

$$= x_1 p_2 - x_2 p_1$$

$$= J_3$$

Similarly, work out $\{J_3, J_2\}$ $\{J_2, J_3\}$ and define

$$\{J_a, J_b\} = \varepsilon_{abc} J_c$$

Algebra of Poisson brackets for angular momentum closes on itself.

Dirac's quantisation $\underline{J} \to \hat{\underline{J}}$ and $[\hat{J}_a, \hat{J}_b] = i\hbar\, \varepsilon_{abc} \hat{J}_c$ and $[\underline{J}] = [\text{action}] = [\hbar]$